AIT-BUDAPEST

AQUINCUM INSTITUTE OF TECHNOLOGY

Combinatorial Optimization

Sample Midterm Test

Instructions

You have 100 minutes to work on the following problems. You may use a calculator but only for elementary calculations; that is, please do not use your calculator for graphing or carrying out programs.

Questions 1 to 5 are marked out of 10. Furthermore, you can collect bonus points by the two bonus problems; these are worth 5 points each. However, solving the bonus problems is optional: the $5 \times 10 = 50$ points achievable by questions 1 to 5 corresponds to a 100 % result.

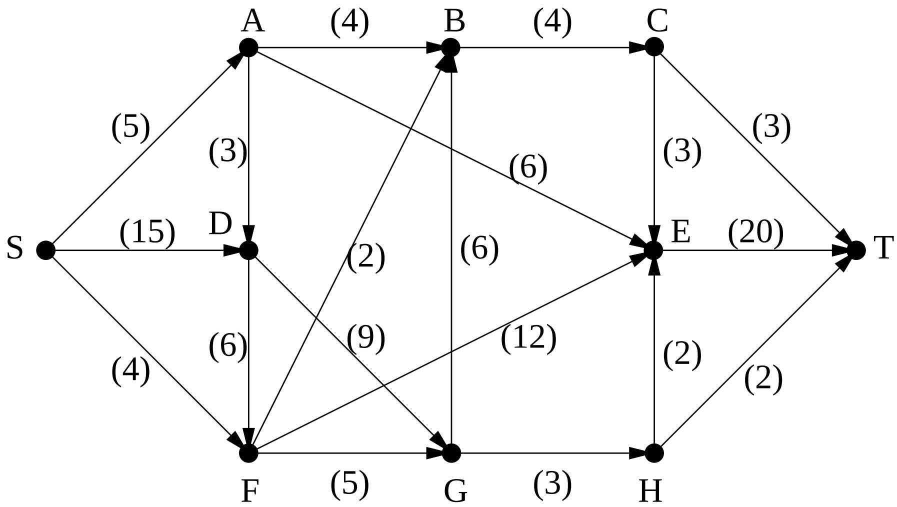
Please make sure to justify your answers in detail. Please write down everything (including calculations, references to material covered, etc) that documents the process of your solution.

Please work in this booklet and only use extra pages if necessary.

1. 8 girls (denoted by A, B, ..., H) and 8 boys (numbered from 1 to 8) in a tribal village have reached the age of getting married. The headman collected information on who is willing to marry whom; his findings are summarized in the table below. Use the augmenting path algorithm to decide if there exists a matching of all 16 youngsters that respects their preferences. Start from the following initial matching: A-1, B-2, C-3, D-4, E-5, F-6.

	A	B	C	D	E	F	G	H
1	♥		♥			♥		
2		♥		♥				
3	♥		♥			♥		
4		♥		♥			♥	
5				♥	♥		♥	♥
6					♥	♥		♥
7	♥		♥					
8	♥	♥	♥					

2. Find a maximum flow (from S to T) and a minimum cut in the following network!



3. SpellCheck Ltd produces two kinds of witches: the WitchOne and the Hag. The production of a WitchOne requires 6 minutes of awfulizing, 8 minutes of uglifying and 1 minute of packaging. The production of a Hag requires 12 minutes of awfulizing, 4 minutes of uglifying and 1 minute of packaging. SpellCheck has 1800 minutes of production time available in its Awfulizing Department, 1400 minutes of production time available in its Uglifying Department and 200 minutes of production time in its Packaging Department each day. A WitchOne sells for \$10, a Hag sells for \$12. Given the above limitations, SpellCheck wants to maximize its profit. Formulate SpellCheck's problem as a linear program and solve it!

4. A student wants to plan his next semester. Besides others, he has a list of seven courses (lettered from A to G) he needs to make up his mind about. He rated the courses according to their interestingness on a scale of 1=boring to 3=exciting and he decided that, no matter what, the average interestingness level of the courses taken will be at least 2. The ratings and the credit values of the courses are shown in the following table:

Course	A	B	C	D	E	F	G
Interestingness rate	2	2	3	1	1	3	3
Number of credits	4	2	3	3	4	2	3

Due to capacity bounds, the student can only take at most four out of the seven courses. Furthermore, he needs to take at least one of the courses D and F. Taking C is only possible if he also takes B. Finally, since A, E and G meet at the same time, he can only take at most one of these three courses.

The student wants to decide which courses to take such that the above requirements are fulfilled and the total number of credits earned by the chosen courses is maximized. Develop an integer programming model for solving the student's problem.

5. Let

$$A = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 3 & 1 \\ -1 & -1 & -2 \\ 0 & -1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} -4 \\ 3 \\ -4 \\ 0 \end{pmatrix}.$$

Use the Fourier-Motzkin elimination to show that the system of linear inequalities $Ax \le b$ is unsolvable and to provide a row vector y that proves this fact according to the Farkas-lemma.

1. Bonus problem. A little bug lives in each vertex of a simple graph G. On a certain day, each bug sets off and moves along an edge to an adjacent vertex of the graph. The bugs want to organize their movement in such a way that finally each vertex will again be occupied by a single bug. Prove that this is possible if and only if no matter how k vertices are deleted from G (where $k \geq 0$ is arbitrary), the resulting graph has at most k isolated vertices. (An isolated vertex is a vertex with no incident edges.)

2. Bonus problem. Let A be an $m \times n$ matrix and b a column vector of dimension m. Prove that exactly one of the following two systems is solvable:

(1) $Ax = b$, $x > 0$

(2) $yA \geq 0$, $yb \leq 0$, $y(A|b) \neq 0$